2020(SPRING)
Problem Sheet MS 566/MI 566 Fourier Analysis

1. Prove that the solution to the Dirichlet's problem is unique.

2. Let $f \in L^1(T)$ and $g \in L^\infty(T)$. Show that
$$\lim_{n\to\infty} \int f(t)g(nt)dt = \widehat{f}(0)\widehat{g}(0).$$

3. Is the solution to the heat equation
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, u(0,x) = f(x), u(t,0) = 0 = u(t,2\pi), t > 0$$
periodic in x?

4. Prove that the Fourier series of a continuously differentiable function f on the circle T is absolutely convergent.

5. Give an example of a series that is Cesaro summable but not convergent.

6. Find the value of
$$\sum_{n=}^{\infty} \frac{1}{(2n+1)^6}.$$

7. Consider the 2π- periodic odd function defined on $[0,\pi]$ by $f(\theta) = \theta(\pi - \theta)$.Compute the Fourier coefficient of f and show that
$$f(\theta) = \frac{8}{\pi} \sum_{k\ odd \geq 1} \frac{\sin k\theta}{k^3}.$$

8. Let $f \in L^1(\mathbb{T})$ and assume $\widehat{f}(0) = 0$ and define
$$F(t) = \int_0^t f(u)du.$$
Show that F is continuous , 2π periodic and
$$\widehat{F}(n) = \frac{1}{in}\widehat{f}(n).$$

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9. Let Let $f, g \in L^1(\mathbb{T})$. Prove that $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$.

10. Assume $f \in L^1(\mathbb{T})$ and $\widehat{f}(n) = O(|\ n\ |^{-k})$. Show that f is m times differentiable and $f^{(m)} \in L^2(\mathbb{T})$ if $k - m > 1/2$.

11. Let $f \in L^1(\mathbb{T})$. Show that
$$\lim_{|n|\to\infty} \widehat{f}(n) = 0.$$

12. Define Fejer kernel F_N. Show that all $N \geq 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(t)dt = 1$$

13. Prove that if a series of complex numbers $\sum c_n$ converges to s, then $\sum c_n$ is Cesaro summable to s.

14. Define Cesaro and Abel summability of an infinite series.Give an example of a series with justification which is abel summable but not cesaro summable.

15. Compute the fourier tranform of the characteristic function of an interval.Let $g_n = \chi_{[-n,n]}$ and $h = \chi_{[-1,1]}$. Find $g_n * h$. Show that $g_n * h$ is the Fourier transform of a function $f_n \in L^1$, except for a multiplicative constant ,

$$f_n = \frac{\sin x \sin nx}{x^2}.$$

Show that $\| f_n \|_1 \to \infty$ and conclude that the mapping $f \mapsto \widehat{f}$ maps L^1 into a proper subset of C_0.

16. If $g(x) = -ixf(x)$ and $g \in L^1$, then show that $\widehat{f}$ is differentiable and $\widehat{f}'(t) = \widehat{g}(t)$.

17. Find the Fourier transform of $e^{-\pi x^2}$.

18. Let G be a compact proper subgroup of T. Show that G is finite and discrete.

19. Let $f \in L^1(T)$ and $g \in L^\infty(T)$. Show that

$$\lim_{n\to\infty} \int f(t)g(nt)dt = \widehat{f}(0)\widehat{g}(0).$$

20. Let f be an integrable function on T such that $\widehat{f}(n) = 0 \; \forall n \in \mathbb{N}$. Show that $f = 0$ at all points of continuity of f.

21. Prove that the Fourier series of a continuously differentiable function f on the circle is absolutely convergent.

22. Let $\{k_n\}$ be a sequence of good kernels and f be an integrable function on T, then show that

$$\lim_{n\to\infty} k_n * f(x) = f(x)$$

whenever f is continuous at x.

23. Let $f \in L^1(\mathbb{R})$. Show that $\widehat{f}$ is uniformly continuous in $-\infty < \xi < \infty$.

24. Show that the space of functions whose Fourier transforms have compact support forms a dense subspace of $L^p(\mathbb{R}), 1 \leq p \leq 2$(Hint: Use Fejer kernel).

25. Suppose that $1 \leq p < r \leq \infty$. Prove that $L^p(T) \supset L^r(T)$.

26. Let f and g be two integrable functions on $[-\pi, \pi]$.

    (a) Define $f * g$.

    (b) Show that $f * g$ is an integrable function.

(c) If either f or g are continuous function, then show that $f * g$ is continuous function.

(d) If either $f \in L^2(\mathbb{T})$ or $g \in L^2(\mathbb{T})$, then show that $f * g \in L^2(\mathbb{T})$.

27. Let f be a continuous function on T. Prove that $\lim_{|n| \to \infty} \widehat{f}(n) = 0$.

28. Find the value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

29. Prove that $\{e^{-int}\}$ forms a complete orthonormal basis for $L^2(T)$.

30. Let f be a C^1 function on $[0, \pi]$ with $f(0) = 0 = f(\pi)$. Prove that

$$\int_0^\pi |f|^2 dt \leq \int_0^\pi |f'|^2 dt.$$

31. Let $f \in L^1(\mathbb{R})$. Find the range of $\mathcal{F}f$, where $\mathcal{F}f = \widehat{f}$.

32. Find $\widehat{f}$ if $f = \chi_{[-1,1]}$.

33. Define Schwartz space. Show that it is closed with respect to the Fourier transformation.

34. Show that there exists unique isometry

$$P : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

which is onto and $P(f) = \widehat{f}, \forall f \in S(\mathbb{R})$.

35. Define tempered distribution.Give the definition of Fourier transform of an $L^P(\mathbb{R})$ function as tempered function. Show that $\widehat{1} = \delta$ and $\widehat{\delta} = 1$.

36. Show that f is analytic on T if and only if there exists constant $k > 0$ and $a > 0$ such that

$$|\widehat{f}(j)| \leq k e^{-a|j|}.$$

Hence show that f is analytic on T if and only if

$$\sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ijz}$$

converges for $|Im(z)| < a$ for some $a > 0$.

37. Show that

$$F(z) = \sum_{n=1}^{\infty} 2^{-n} \frac{1}{(z+n)^2 + n^{-1}}$$

is analytic on $\mathbb{R}$ and $F|_{\mathbb{R}} \in L^1 \bigcap L^\infty(\mathbb{R})$, however F is not holomorphic in any strip $\{z : |y| < a\}, a > 0$. (Hint: Use Paley-Wiener Theorem)

38. Show that

$$F(z) = e^{-e^{z^2}}$$

is entire. $F|_{\mathbb{R}} \in L^1 \bigcap L^\infty(\mathbb{R})$, however f is unbounded on any line $y = constant \neq 0$.

39. State the Heisenberg uncertainty principle in Fourier analysis. When does the equality hold?

40. Write down the probability of a particle located in the interval (a,b).

41. Show that the Fourier transform of a radial function is radial.

42. Write the relation between the Fourier transform and Radon transform of a Schwartz class (S) of function.Use this to show $R(f) = R(g)$ implies $f = g$ for any $f, g \in S$ where R denotes the Radon transform.

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